

# REGULARITY OF NONLOCAL MINIMAL CONES IN DIMENSION 2

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ABSTRACT. We show that the only nonlocal  $s$ -minimal cones in  $\mathbb{R}^2$  are the trivial ones for all  $s \in (0, 1)$ . As a consequence we obtain that the singular set of a nonlocal minimal surface has at most  $n - 3$  Hausdorff dimension.

## 1. INTRODUCTION

Nonlocal minimal surfaces were introduced in [2] as boundaries of measurable sets  $E$  whose characteristic function  $\chi_E$  minimizes a certain  $H^{s/2}$  norm. More precisely, for any  $s \in (0, 1)$ , the nonlocal  $s$ -perimeter functional  $\text{Per}_s(E, \Omega)$  of a measurable set  $E$  in an open set  $\Omega \subset \mathbb{R}^n$  is defined as the  $\Omega$ -contribution of  $\chi_E$  in  $\|\chi_E\|_{H^{s/2}}$ , that is

$$(1) \quad \text{Per}_s(E, \Omega) := L(E \cap \Omega, \mathbb{R}^n \setminus E) + L(E \setminus \Omega, \Omega \setminus E),$$

where  $L(A, B)$  denotes the double integral

$$L(A, B) := \int_A \int_B \frac{dx dy}{|x - y|^{n+s}}, \quad A, B \text{ measurable sets.}$$

A set  $E$  is  $s$ -minimal in  $\Omega$  if  $\text{Per}_s(E, \Omega)$  is finite and

$$\text{Per}_s(E, \Omega) \leq \text{Per}_s(F, \Omega)$$

for any measurable set  $F$  for which  $E \setminus \Omega = F \setminus \Omega$ .

We say that  $E$  is  $s$ -minimal in  $\mathbb{R}^n$  if it is  $s$ -minimal in any ball  $B_R$  for any  $R > 0$ . The boundary of  $s$ -minimal sets are referred to as *nonlocal  $s$ -minimal surfaces*.

The theory of nonlocal minimal surfaces developed in [2] is (at least for some features) similar to the theory of standard minimal surfaces. In fact as  $s \rightarrow 1^-$ , the  $s$ -minimal surfaces converge to the classical minimal surfaces and the functional in (1) (after a multiplication by a factor of the order of  $(1 - s)$ ) Gamma-converges to the classical perimeter functional (see [3, 1]).

In [2] it was shown that nonlocal  $s$ -minimal surfaces are  $C^{1,\alpha}$  outside a singular set of Hausdorff dimension  $n - 2$ . The precise dimension of the singular set is determined by the problem of existence in low dimensions of a nontrivial global  $s$ -minimal cone (i.e. an  $s$ -minimal set  $E$  such that  $tE = E$  for any  $t > 0$ ). In the case of classical minimal surfaces Simons theorem states that the only global minimal cones in dimension  $n \leq 7$  must be half-planes, which implies that the Hausdorff dimension of the singular set of a minimal surface in  $\mathbb{R}^n$  is  $n - 8$ . In [4], the authors used these results to show that if  $s$  is sufficiently close to 1 the same holds for  $s$ -minimal surfaces i.e. global  $s$ -minimal cones must be half-planes if  $n \leq 7$  and the Hausdorff dimension of the singular set is  $n - 8$ .

Given the nonlocal character of the functional in (1), it seems more difficult to analyze global  $s$ -minimal cones for general values of  $s \in (0, 1)$ . The purpose of this paper is to show that there are no nontrivial  $s$ -minimal cones in the plane. Our theorem is the following.

**Theorem 1.** *If  $E$  is an  $s$ -minimal cone in  $\mathbb{R}^2$ , then  $E$  is a half-plane.*

From Theorem 1 above and Theorem 9.4 of [2], we obtain that  $s$ -minimal sets in two-dimensional domains are locally  $C^{1,\alpha}$ . Also, from Theorem 1 and classical blow-up and blow-down arguments<sup>1</sup>, we obtain that  $s$ -minimal sets in the plane are half-planes. We summarize these observations in the following result:

**Corollary 1.** *If  $E$  is an  $s$ -minimal set in  $\Omega \subset \mathbb{R}^2$ , and  $\Omega' \Subset \Omega$ , then  $(\partial E) \cap \Omega'$  is a  $C^{1,\alpha}$ -curve. If  $E$  is an  $s$ -minimal set in  $\mathbb{R}^2$ , then  $\partial E$  is a straight line.*

In higher dimensions, by combining the result of Theorem 1 here with the dimensional reduction performed in [2], we obtain that any nonlocal  $s$ -minimal surface in  $\mathbb{R}^n$  is locally  $C^{1,\alpha}$  outside a singular set of Hausdorff dimension  $n - 3$ .

**Corollary 2.** *Let  $\partial E$  be a nonlocal  $s$ -minimal surface in  $\Omega \subset \mathbb{R}^n$  and let  $\Sigma_E \subset \partial E \cap \Omega$  denote its singular set. Then  $\mathcal{H}^d(\Sigma_E) = 0$  for any  $d > n - 3$ .*

The idea of the proof of Theorem 1 is the following. If  $E \subset \mathbb{R}^2$  is an  $s$ -minimal cone then we construct a set  $\tilde{E}$  as a translation of  $E$  in  $B_{R/2}$  which coincides with  $E$  outside  $B_R$ . Then the difference between the energies (of the extension) of  $\tilde{E}$  and  $E$  tends to 0 as  $R \rightarrow \infty$ . This implies that also the energy of  $E \cap \tilde{E}$  is arbitrarily close to the energy of  $E$ . On the other hand if  $E$  is not a half-plane the set  $\tilde{E} \cap E$  can be modified locally to decrease its energy by a fixed small amount and we reach a contradiction.

In the next section we introduce some notation and obtain the perturbative estimates that are needed for the proof of Theorem 1 in Section 3.

## 2. PERTURBATIVE ESTIMATES

We start by introducing some notation.

### Notation.

We denote points in  $\mathbb{R}^n$  by lower case letters, such as  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and points in  $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, +\infty)$  by upper case letters, such as  $X = (x, x_{n+1}) = (x_1, \dots, x_{n+1}) \in \mathbb{R}_+^{n+1}$ .

The open ball in  $\mathbb{R}^{n+1}$  of radius  $R$  and center 0 is denoted by  $B_R$ . Also we denote by  $B_R^+ := B_R \cap \mathbb{R}_+^{n+1}$  the open half-ball in  $\mathbb{R}^{n+1}$  and by  $S_+^n := S^n \cap \mathbb{R}_+^{n+1}$  the unit half-sphere.

The fractional parameter  $s \in (0, 1)$  will be fixed throughout this paper; we also set

$$a := 1 - s \in (0, 1).$$

The standard Euclidean base of  $\mathbb{R}^{n+1}$  is denoted by  $\{e_1, \dots, e_{n+1}\}$ . Whenever there is no possibility of confusion we identify  $\mathbb{R}^n$  with the hyperplane  $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ .

The transpose of a square matrix  $A$  will be denoted by  $A^T$ , and the transpose of a row vector  $V$  is the column vector denoted by  $V^T$ . We denote by  $I$  the identity matrix in  $\mathbb{R}^{n+1}$ .

We introduce the functional

$$(2) \quad \mathcal{E}_R(u) := \int_{B_R^+} |\nabla u(X)|^2 x_{n+1}^a dX.$$

which is related to the  $s$ -minimal sets by an extension problem, as shown in Section 7 of [2]. More precisely, given a set  $E \subseteq \mathbb{R}^n$  with locally finite  $s$ -perimeter, we can associate to it uniquely its extension function  $u : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$  whose trace on  $\mathbb{R}^n \times \{0\}$  is given by  $\chi_E - \chi_{\mathbb{R}^n \setminus E}$  and which minimizes the energy functional in (2) for any  $R > 0$ .

<sup>1</sup>For instance, one can use the proof of Theorem III.8.17 in [5], where the density estimates, the compactness arguments and the monotonicity formulas for classical minimal surfaces are replaced by the ones in [2].

Of course, in all the results presented, we are implicitly ruling out the trivial case in which either the  $s$ -minimal set  $E$  or its complement is empty.

We recall (see Proposition 7.3 of [2]) that  $E$  is  $s$ -minimal in  $\mathbb{R}^n$  if and only if its extension  $u$  is minimal for the energy in (2) under compact perturbations whose trace in  $\mathbb{R}^n \times \{0\}$  takes the values  $\pm 1$ . More precisely, for any  $R > 0$ ,

$$(3) \quad \mathcal{E}_R(u) \leq \mathcal{E}_R(v)$$

for any  $v$  that coincides with  $u$  on  $\partial B_R^+ \cap \{x_{n+1} > 0\}$  and whose trace on  $\mathbb{R}^n \times \{0\}$  is given by  $\chi_F - \chi_{\mathbb{R}^n \setminus F}$  for any measurable set  $F$  which is a compact perturbation of  $E$  in  $B_R$ .

Next we estimate the variation of the functional in (2) with respect to horizontal domain perturbations. For this we introduce a standard cutoff function

$$\varphi \in C_0^\infty(\mathbb{R}^{n+1}), \text{ with } \varphi(X) = 1 \text{ if } |X| \leq 1/2 \text{ and } \varphi(X) = 0 \text{ if } |X| \geq 3/4.$$

Given  $R > 0$ , we let

$$(4) \quad Y := X + \varphi(X/R)e_1.$$

Then we have that  $X \mapsto Y = Y(X)$  is a diffeomorphism of  $\mathbb{R}_+^{n+1}$  as long as  $R$  is sufficiently large (possibly in dependence of  $\varphi$ ).

Given a measurable function  $u : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$ , we define

$$(5) \quad u_R^+(Y) := u(X).$$

Similarly, by switching  $e_1$  with  $-e_1$  (or  $\varphi$  with  $-\varphi$  in (4)), we can define  $u_R^-(Y)$ .

In the next lemma we estimate a discrete second variation for the energy  $\mathcal{E}_R(u)$ .

**Lemma 1.** *Suppose that  $u$  is homogeneous of degree zero and  $\mathcal{E}_R(u) < +\infty$ . Then*

$$(6) \quad |\mathcal{E}_R(u_R^+) + \mathcal{E}_R(u_R^-) - 2\mathcal{E}_R(u)| \leq CR^{n-3+a},$$

for a suitable  $C \geq 0$ , depending on  $\varphi$  and  $u$ .

*Proof.* We start with the following observation. Let us consider the square matrix of order  $(n+1)$

$$A := \begin{pmatrix} a_1 & \dots & \dots & a_{n+1} \\ 0 & \dots & \dots & 0 \\ & \ddots & & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

with  $1 + a_1 \neq 0$ . Then a direct computation shows that

$$(7) \quad (I + A)^{-1} = I - \frac{1}{1 + a_1}A = I - \frac{A}{\det(I + A)}.$$

Now, we define

$$\chi_R(X) := \begin{cases} 1 & \text{if } R/2 \leq |X| \leq R, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}(X) := \frac{1}{R} \begin{pmatrix} \partial_1 \varphi(X/R) & \dots & \dots & \partial_{n+1} \varphi(X/R) \\ 0 & \dots & \dots & 0 \\ & & \ddots & \\ 0 & \dots & \dots & 0 \end{pmatrix}.$$

Notice that

$$(8) \quad \mathcal{M} = O(1/R) \chi_R.$$

Let now

$$\kappa(X) := |\det D_X Y(X)| = \det(I + \mathcal{M}(X)) = 1 + \frac{\partial_1 \varphi(X/R)}{R} = 1 + \operatorname{tr} \mathcal{M}(X).$$

By (7), we see that

$$(9) \quad (D_X Y)^{-1} = (I + \mathcal{M})^{-1} = I - \frac{\mathcal{M}}{\kappa}.$$

Also,  $1/\kappa = 1 + O(1/R)$ , therefore, by (8),

$$(10) \quad \frac{\mathcal{M} \mathcal{M}^T}{\kappa} = O(1/R^2) \chi_R.$$

Now, we perform some chain rule differentiation of the domain perturbation. For this, we take  $X$  to be a function of  $Y$ ; also, the functions  $u$ ,  $Y$ ,  $\chi_R$ ,  $\mathcal{M}$  and  $\kappa$  will be evaluated at  $X$ , while  $u_R^+$  will be evaluated at  $Y$  (e.g., the row vector  $\nabla_X u$  is a short notation for  $\nabla_X u(X)$ , while  $\nabla_Y u_R^+$  stands for  $\nabla_Y u_R^+(Y)$ ). We use (5) and (9) to obtain

$$\nabla_Y u_R^+ = \nabla_X u D_Y X = \nabla_X u (D_X Y)^{-1} = \nabla_X u \left( I - \frac{\mathcal{M}}{\kappa} \right).$$

Also, by changing variables,

$$dY = |\det D_X Y| dX = \kappa dX.$$

Accordingly

$$\begin{aligned} |\nabla_Y u_R^+|^2 y_{n+1}^a dY &= \nabla_X u \left( I - \frac{\mathcal{M}}{\kappa} \right) \left( I - \frac{\mathcal{M}}{\kappa} \right)^T (\nabla_X u)^T x_{n+1}^a \kappa dX \\ &= \nabla_X u \left( \kappa I - \mathcal{M} - \mathcal{M}^T + \frac{\mathcal{M} \mathcal{M}^T}{\kappa} \right) (\nabla_X u)^T x_{n+1}^a dX \\ &= \nabla_X u \left( (1 + \operatorname{tr} \mathcal{M}) I - \mathcal{M} - \mathcal{M}^T + \frac{\mathcal{M} \mathcal{M}^T}{\kappa} \right) (\nabla_X u)^T x_{n+1}^a dX. \end{aligned}$$

Hence, from (10),

$$\begin{aligned} |\nabla_Y u_R^+|^2 y_{n+1}^a dY &= \nabla_X u \left( (1 + \operatorname{tr} \mathcal{M}) I - \mathcal{M} - \mathcal{M}^T + O(1/R^2) \chi_R \right) (\nabla_X u)^T x_{n+1}^a dX. \end{aligned}$$

The similar term for  $\nabla_Y u_R^-$  may be computed by switching  $\varphi$  to  $-\varphi$  (which makes  $\mathcal{M}$  switch to  $-\mathcal{M}$ ): thus we obtain

$$\begin{aligned} |\nabla_Y u_R^-|^2 y_{n+1}^a dY &= \nabla_X u \left( (1 - \operatorname{tr} \mathcal{M}) I + \mathcal{M} + \mathcal{M}^T + O(1/R^2) \chi_R \right) (\nabla_X u)^T x_{n+1}^a dX. \end{aligned}$$

By summing up the last two expressions, after simplification we conclude that

$$(11) \quad \left( |\nabla_Y u_R^+|^2 + |\nabla_Y u_R^-|^2 \right) y_{n+1}^a dY = 2 \left( 1 + O(1/R^2) \chi_R \right) |\nabla_X u|^2 x_{n+1}^a dX.$$

On the other hand, the function  $g(X) := |\nabla_X u(X)|^2 x_{n+1}^a$  is homogeneous of degree  $a - 2$ , hence

$$\begin{aligned} \int_{B_R^+} \chi_R |\nabla_X u|^2 x_{n+1}^a dX &= \int_{B_R^+ \setminus B_{R/2}^+} g dX = \int_{R/2}^R \left[ \int_{S_+^n} g(\vartheta \varrho) d\vartheta \right] \varrho^n d\varrho \\ &= \int_{R/2}^R \varrho^{n+a-2} \left[ \int_{S_+^n} g(\vartheta) d\vartheta \right] d\varrho = CR^{n+a-1}, \end{aligned}$$

for a suitable  $C \geq 0$  depending on  $u$ . This and (11) give that

$$\begin{aligned} \int_{B_R^+} \left( |\nabla_Y u_R^+|^2 + |\nabla_Y u_R^-|^2 \right) y_{n+1}^a dY - 2 \int_{B_R^+} |\nabla_X u|^2 x_{n+1}^a dX \\ = O(1/R^2) \int_{B_R^+} \chi_R |\nabla_X u|^2 x_{n+1}^a dX \\ = O(1/R^2) \cdot CR^{n+a-1}, \end{aligned}$$

which completes the proof of the lemma.  $\square$

Lemma 1 turns out to be particularly useful when  $n = 2$ . In this case (6) yields

$$(12) \quad \mathcal{E}_R(u_R^+) + \mathcal{E}_R(u_R^-) - 2\mathcal{E}_R(u) \leq \frac{C}{R^s},$$

and the right hand side becomes arbitrarily small for large  $R$ . As a consequence, we also obtain the following corollary.

**Corollary 3.** *Suppose that  $E$  is an  $s$ -minimal cone in  $\mathbb{R}^2$  and that  $u$  is the extension of  $\chi_E - \chi_{\mathbb{R}^2 \setminus E}$ . Then*

$$(13) \quad \mathcal{E}_R(u_R^+) \leq \mathcal{E}_R(u) + \frac{C}{R^s}.$$

*Proof.* Since  $E$  is a cone, we know that  $u$  is homogeneous of degree zero (see Corollary 8.2 in [2]): thus, the assumptions of Lemma 1 are fulfilled and so (12) holds true.

From the minimality of  $u$  (see (3)), we infer that

$$\mathcal{E}_R(u) \leq \mathcal{E}_R(u_R^-),$$

which together with (12) gives the desired claim.  $\square$

### 3. PROOF OF THEOREM 1

We argue by contradiction, by supposing that  $E \subset \mathbb{R}^2$  is an  $s$ -minimal cone different than a half-plane. By Theorem 10.3 in [2],  $E$  is the disjoint union of a finite number of closed sectors. Then, up to a rotation, we may suppose that a sector of  $E$  has angle less than  $\pi$  and is bisected by  $e_2$ . Thus, there exist  $M \geq 1$  and  $p \in B_M$ , on the  $e_2$ -axis, such that  $p$  lies in the interior of  $E$ , and  $p + e_1$  and  $p - e_1$  lie in the exterior of  $E$ .

Let  $R > 4M$  be sufficiently large. Using the notation of Lemma 1 we have

$$(14) \quad \begin{aligned} u_R^+(Y) &= u(Y - e_1), \text{ for all } Y \in B_{2M}^+, \text{ and} \\ u_R^+(Y) &= u(Y) \text{ for all } Y \in \mathbb{R}_+^3 \setminus B_R^+, \end{aligned}$$

where  $u$  is the extension of  $\chi_E - \chi_{\mathbb{R}^2 \setminus E}$ . We define

$$v_R(X) := \min\{u(X), u_R^+(X)\} \quad \text{and} \quad w_R(X) := \max\{u(X), u_R^+(X)\}.$$

Denote  $P := (p, 0) \in \mathbb{R}^3$ . We claim that

$$(15) \quad \begin{aligned} u_R^+ &< w_R = u \text{ in a neighborhood of } P, \text{ and} \\ u &< w_R = u_R^+ \text{ in a neighborhood of } P + e_1. \end{aligned}$$

Indeed, by (14)

$$u_R^+(P) = u(P - e_1) = (\chi_E - \chi_{\mathbb{R}^2 \setminus E})(p - e_1) = -1$$

while

$$u(P) = (\chi_E - \chi_{\mathbb{R}^2 \setminus E})(p) = 1.$$

Similarly,  $u_R^+(P + e_1) = u(P) = 1$  while  $u(P + e_1) = -1$ . This and the continuity of the functions  $u$  and  $u_R^+$  at  $P$ , respectively  $P + e_1$ , give (15).

We point out that  $\mathcal{E}_R(u) \leq \mathcal{E}_R(v_R)$ , thanks to (14) and the minimality of  $u$ . This and the identity

$$\mathcal{E}_R(v_R) + \mathcal{E}_R(w_R) = \mathcal{E}_R(u) + \mathcal{E}_R(u_R^+)$$

imply that

$$(16) \quad \mathcal{E}_R(w_R) \leq \mathcal{E}_R(u_R^+).$$

Now we observe that  $w_R$  is not a minimizer for  $\mathcal{E}_{2M}$  with respect to compact perturbations in  $B_{2M}^+$ . Indeed, if  $w_R$  were a minimizer we use  $u \leq w_R$  and the first fact in (15) to conclude  $u = w_R$  in  $B_{2M}^+$  from the strong maximum principle. However this contradicts the second inequality in (15).

Therefore, we can modify  $w_R$  inside a compact set of  $B_{2M}^+$  and obtain a competitor  $u_*$  such that

$$\mathcal{E}_{2M}(u_*) + \delta \leq \mathcal{E}_{2M}(w_R),$$

for some  $\delta > 0$ , independent of  $R$  (since  $w_R$  restricted to  $B_{2M}^+$  is independent of  $R$ , by (14)).

The inequality above implies

$$(17) \quad \mathcal{E}_R(u_*) + \delta \leq \mathcal{E}_R(w_R),$$

since  $u_*$  and  $w_R$  coincide outside  $B_{2M}^+$ . Thus, we use (13), (16) and (17) to conclude that

$$\mathcal{E}_R(u_*) + \delta \leq \mathcal{E}_R(w_R) \leq \mathcal{E}_R(u_R^+) \leq \mathcal{E}_R(u) + \frac{C}{R^s}.$$

Accordingly, if  $R$  is large enough we have that  $\mathcal{E}_R(u_*) < \mathcal{E}_R(u)$ , which contradicts the minimality of  $u$ . This completes the proof of Theorem 1.

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